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VARIATIONAL PRINCIPLES IN DYNAMIC THERMOVISCOELASTICITY†

SUBRATA MUKHERJEE‡

Department of Applied Mechanics, Stanford University, Stanford, California 94305

Abstract—Dual variational principles for steady state wave propagation in three dimensional thermoviscoelastic media are presented. The first one, for the equations of motion, involves only the complex displacement function. The second principle is for the energy equation. These principles are specialized to the case of one dimension. A one-dimensional example, that of wave propagation in a thermoviscoelastic rod insulated on its lateral surface and driven by a sinusoidal stress at one end, is solved using the Rayleigh–Ritz method. The displacement and temperature functions are expressed as series of polynomials. Successive approximations for the solution are compared with a solution obtained by a method of finite differences, and an estimate of the degree of accuracy as a function of the number of terms taken in the series is obtained. It is found that the approximate solution converges rapidly to the correct one.

INTRODUCTION

SEVERAL authors have developed and used variational principles to obtain solutions to problems in quasi-static and dynamic viscoelasticity, with and without thermomechanical coupling. Gurtin [1] and Leitman [2] have developed variational principles for viscoelastic media without thermomechanical coupling. They have used the convolution form of the constitutive equations and have developed variational principles for several types of boundary value problems. Their work appears to be primarily of mathematical interest. Schapery [3, 4] also gives convolution variational principles with and without thermomechanical coupling respectively. Valanis [5] has developed a principle applicable to viscoelastic materials with constant Poisson's ratio, without thermomechanical coupling.

Schapery [8, 9] has studied vibration of viscoelastic media with thermomechanical coupling. In [9] he has used a complex modulus form of the constitutive equations and has developed a variational principle analogous to Reissner's complementary principle using complex kinetic and potential "energy" functions. His principle involves both stress and displacement functions which must *a priori* satisfy the equations of motion. He has considered examples with bodies that are either massless or with concentrated mass, and in his last example of a "solid cylinder with distributed mass" he only gives a first approximation to the solution, using only one term of a series expansion. While his method appears promising, the question of convergence to the exact solution, or, in other words, how many terms in the series are necessary to get a sufficiently close approximation to the exact solution, remains open.

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‡ Graduate Student and Research Assistant: currently Consultant, Cartridge Television Inc. 1080 North 7th. Street, San Jose, California 95112.

This paper is concerned with the application of variational principles to problems of steady state wave propagation in viscoelastic media with thermomechanical coupling. A complex modulus description of the constitutive equations is used. The material is assumed to be thermorheologically simple [7] and the energy equation, as suggested by Schapery [9], uses the cycle averaged temperature distribution with the cycle averaged value of the Rayleigh dissipation function acting as the heat source. The displacement variational principle suggested here involves only the complex displacement function and an admissible set of displacement functions need only satisfy any prescribed displacement boundary conditions that might exist. This principle can be considered to be an extension of that developed by Kohn, Krumhansl and Lee [6] for elastic media. It uses complex instead of real "energy" functions. The temperature variational principle is the one suggested by Biot [10] and Schapery [9].

An alternative form of these principles is suggested. This proves more useful for certain applications. These principles are set up for general three-dimensional problems and are later specialized to the case of one dimension.

As an example, the problem of steady state longitudinal waves in a viscoelastic rod with thermal coupling subjected to a sinusoidal stress applied at one end, is solved using a variational approach. Huang and Lee [11] solved this problem including time as an independent variable. This resulted in partial differential equations which were solved numerically using a method of finite differences. This is useful if the time histories of the stress and temperature have to be determined. For many engineering design applications, however, the steady state values of stress and temperature are of primary interest, since due to dissipation of mechanical energy the temperature increases until a steady state is reached, if in fact the situation is stable. Such a steady state yields the most severe temperature conditions which are the major concern in design. In such cases it is simpler and far more efficient to obtain the steady state values directly instead of following the complete time history of the process till a steady state is reached. In this example, the steady state values of stress and temperature have been directly obtained by using a Rayleigh–Ritz procedure on the alternative form of the variational principles. Functions for displacement (complex) and temperature are assumed as polynomial series (for convenience) with "n" and "m" terms respectively, with unknown coefficients. Simultaneous extremization of two functionals is carried out by solving the resultant nonlinear algebraic equations in a computer. The number of terms "n" and "m" can be set in the program. Calculations for a Lockheed solid propellant [8, 11] are carried out for various values for "n" and "m" and the question of rapidity of convergence to the solution given in [11] is discussed.

GOVERNING DIFFERENTIAL EQUATIONS

1. *General equations in three dimensions*

Let us consider the governing differential equations for stresses, displacements and temperature in steady state oscillations of linear isotropic viscoelastic media. The thermomechanical coupling is caused by the cycle averaged value of the mechanical dissipation function acting as the heat source in the energy equation and by the fact that the complex viscoelastic modulii are temperature dependent. As pointed out by Schapery [8, 9] the coupling terms due to the dilatation and potential energy drop out of the energy equation if it is integrated over a cycle.

We assume steady state conditions where the mechanical variables are harmonic functions of time, and the temperature, after a sufficiently long time, is independent of time. Strictly speaking, the temperature is never truly time independent but has small cyclic variations about a mean value as a result of the cyclic variations of the potential energy, dilatation and dissipation (see [11]). These small fluctuations, however, will be neglected and henceforth the temperature will mean its cycle averaged steady state value.

Let the stress and strain tensors and the displacement vector be defined as the real parts of

$$\begin{aligned} \tilde{\sigma}_{ij} &= \sigma_{ij}\,\mathrm{e}^{i\omega t} \\ \tilde{\varepsilon}_{ij} &= \varepsilon_{ij}\,\mathrm{e}^{i\omega t} \\ \tilde{u}_i &= u_i\,\mathrm{e}^{i\omega t} \end{aligned} \tag{1}$$

where $i = \sqrt{(-1)}$, ω is the frequency (real) and t is time. The complex quantities σ_{ij}, ε_{ij} and u_i are used most of the time in further calculations and will be referred to as simply stress, strain and displacement respectively. The familiar cartesian tensor notation is used here. The suffixes i and j range from 1 to 3 and a repeated index implies summation over that index.

The equations of motion are

$$\sigma_{ij,j} + \rho\omega^2 u_i = 0 \tag{2}$$

where ρ is the mass density (assumed constant).

The constitutive equations, using the familiar complex modulus formulation, are

$$\sigma_{ij} = \lambda^* u_{k,k}\delta_{ij} + \mu^*(u_{i,j} + u_{j,i}) \tag{3}$$

where λ^* and μ^* are complex Lamé parameters which are functions of temperature, and δ_{ij} is the Kronecker delta.

The Lamé parameters are related to the more commonly used complex shear and bulk compliances J^* and B^* by the relations

$$\begin{aligned} \mu^* &= \frac{1}{J^*} \\ \lambda^* &= \frac{1}{B^*} - \frac{2}{3J^*}. \end{aligned} \tag{4}$$

Typically, in polymers, for temperatures above the glass transition temperature, J^* is a very strong nonlinear function of temperature while B^* is a relatively weak function of temperature.

In thermorheologically simple materials it is assumed that J^* is a function of only the reduced frequency ω' which is related to the actual frequency ω through the temperature dependent shift factor a_T (see [9]), i.e.

$$\omega' = \omega a_T(T) \tag{5}$$

where a_T represents the effect of temperature on viscosity.

Combining equations (2) and (3) one can write the equations of motion in terms of displacement alone

$$(\lambda^* u_{k,k})_{,i} + (\mu^* u_{i,j})_{,j} + (\mu^* u_{j,i})_{,j} + \rho\omega^2 u_i = 0. \tag{6}$$

The steady state energy equation for the cycle averaged temperature distribution is given by

$$KT_{,ii} = -2D \tag{7}$$

where T is the temperature, K is the thermal conductivity (assumed constant) and D is the cycle averaged value of the Rayleigh dissipation function given by

$$D = \frac{\omega}{4\pi} \int_t^{t+2\pi/\omega} \mathrm{Re}(\tilde{\sigma}_{ij}) \, \mathrm{Re}\left(\frac{\partial \tilde{\varepsilon}_{ij}}{\partial t'}\right) \mathrm{d}t'$$

where Re denotes the real part of the complex argument. $2D$ is the cycle averaged value of the mechanical dissipation.

Using equations (1), (3) and (7) and after carrying out the necessary integration, we have

$$D = \frac{\omega}{4}[\lambda_2|\varepsilon_{kk}|^2 + 2\mu_2|\varepsilon_{ij}|^2] \tag{8}$$

where

$$\lambda^* = \lambda_1 + i\lambda_2$$

$$\mu^* = \mu_1 + i\mu_2$$

λ_1 and μ_1 are the storage Lamé parameters and λ_2, μ_2 the loss parameters,

$$|\varepsilon_{kk}|^2 = \varepsilon_{kk}\bar{\varepsilon}_{jj}$$

$$|\varepsilon_{ij}|^2 = \varepsilon_{ij}\bar{\varepsilon}_{ij}$$

and "—" denotes the complex conjugate.

Note that as defined D is a real function of the strain tensor and the loss parameters λ_2 and μ_2.

Using the familiar kinematic relations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{9}$$

we can write equation (7) in terms of displacement and temperature

$$KT_{,ii} + \frac{\omega}{2}[\lambda_2|u_{k,k}|^2 + \mu_2(u_{i,j}\bar{u}_{i,j} + u_{i,j}\bar{u}_{j,i})] = 0. \tag{10}$$

Equations (6) and (10) are a complete set of four nonlinear partial differential equations for the four unknowns u_i ($i = 1, 3$) and T. Since these are written in terms of displacements, the compatibility conditions are automatically satisfied.

For the boundary conditions we assume that the displacement vector is prescribed on a part of the surface A_u, while the traction vector is prescribed on the remainder A_σ. Also, the temperature is prescribed on the portion of the surface A_T and the heat flux (per unit area) is prescribed on the remaining surface A_H.

VARIATIONAL PRINCIPLES

1. *Variational principles in three dimensions*

The field equations (6) and (10) for the coupled thermomechanical problem together with the boundary conditions are equivalent to two variational principles.

The variational principle for the equations of motion and the constitutive equations (2) and (3) can be stated as: of all displacement functions u_i satisfying prescribed displacements u_i on A_u, the displacement function satisfying the equations of motion (2), the constitutive equations (3) and the traction boundary condition on A_σ is determined by

$$\underset{u}{\delta}\left\{\int_V (U_v - K_v)\,\mathrm{d}V - \int_{A_\sigma} u_i \mathring{\sigma}_{ij} n_j\,\mathrm{d}A\right\} = 0 \tag{11}$$

where U_v and K_v are analogous to the elastic strain energy density and kinetic energy density and are given by

$$U_v = \frac{\lambda^*}{2}(u_{k,k})^2 + \frac{\mu^*}{2}(u_{i,j})(u_{i,j}+u_{j,i})$$

$$K_v = \tfrac{1}{2}\rho\omega^2 u_i u_i$$

$\mathring{\sigma}_{ij}n_j$ = prescribed traction on A_σ

n_j = direction cosines of the outward unit normal to the surface A_σ

$\underset{u}{\delta}$ means that the variations must be taken with respect to the displacement function only.

If the kinematic equations (10) are also satisfied (i.e., we define strain functions to satisfy equation (10)), we can write

$$U_v = \frac{\lambda^*}{2}(\varepsilon_{kk})^2 + \mu^*\varepsilon_{ij}\varepsilon_{ij}.$$

This variational principle is analogous to Hamilton's principle in dynamic elasticity.

Comparing equation (11) with the variational principle given by Schapery in [9] we see that here we can choose trial functions for the displacement which need only satisfy the displacement boundary conditions of the problem whereas in [9] Schapery must choose displacement and stress functions which must *a priori* satisfy the equations of motion. In many cases (e.g. when displacements are prescribed over the entire boundary, i.e. $A_\sigma = 0$) this principle appears more powerful than Schapery's, since at least the same degree of accuracy in the solution can possibly be achieved with an equal number of terms in the two cases, with the definite advantage of not having to satisfy, *a priori*, the equations of motion. In other cases the situation is less clear and further study regarding the relative merits of the two principles seems necessary.

The variational principle can be proved by carrying out variations with respect to the function u_i to yield

$$-\int_V \{(\lambda^* u_{k,k})_{,i} + (\mu^* u_{i,j})_{,j} + (\mu^* u_{j,i})_{,j} + \rho\omega^2 u_i\}\,\delta u_i\,\mathrm{d}V$$

$$+\int_{A_\sigma} \{\lambda^* u_{k,k}\delta_{ij} + \mu^*(u_{i,j}+u_{j,i}) - \mathring{\sigma}_{ij}\}n_j\,\delta u_i\,\mathrm{d}A = 0. \tag{12}$$

In view of the arbitrariness of δu_i, this expression equals zero only if the equations of motion (2), the constitutive equations (3), and the traction boundary conditions on A_σ are satisfied.

If we restrict the admissible class of displacement functions such that the boundary conditions for both the displacements on A_u and traction on A_σ (through equation 3) are satisfied, the surface integral drops out of equation (12) and we are left with a simplified form of the principle.

Equation (12) can be considered to be an alternative form of the variational principle (11).

It is useful to compare the relative advantages of the two forms. Equation (11) uses energy invariants and therefore appears more convenient in complicated coordinate systems. However, when carrying out a Rayleigh–Ritz method of solution, use of equation (12) can save a large amount of calculations since the variations have already been carried out.

It must be remembered that λ^* and μ^* are temperature dependent and in order to get the temperature field we require another variational principle from the energy equation. This can be stated as follows: of all temperature distributions which satisfy prescribed $\mathring{T}$ on A_T, the temperature distribution which also satisfies the energy equation (7) and the heat flow boundary condition on A_H is determined by

$$\underset{T}{\delta}\left\{\int_V (S_T - S_M)\,\mathrm{d}V + \int_{A_H} \mathring{H}T\,\mathrm{d}A\right\} = 0 \tag{13}$$

where S_T is proportional to the entropy production density resulting from temperature gradients (see [7])

$$S_T = \tfrac{1}{2}KT_{,i}T_{,i}$$

and S_M is the integral of the mechanical dissipation

$$S_M = 2\int^T D\,\mathrm{d}T'$$

$\mathring{H}$ = prescribed heat flux per unit area out of the body.

$\underset{T}{\delta}$ means the variations must be taken with respect to the temperature only;

D is as given by equations (7) and (10).

This principle can be proved by taking variations with respect to T to yield

$$-\int_V (2D + KT_{,ii})\,\delta T\,\mathrm{d}V + \int_{A_H} (KT_{,i}n_i + \mathring{H})\,\delta T\,\mathrm{d}A = 0. \tag{14}$$

In view of the arbitrariness of δT, this expression is zero only if the energy equation (10) and the heat flow boundary condition on A_H is satisfied.

If we restrict the admissible class of temperature functions such that the boundary conditions for both the temperature on A_T and heat flow on A_H are satisfied, the surface integral drops out of equation (14). Equation (14) is an alternative form of the variational principle (13). Equation (13) uses thermodynamic invariants and the comments made earlier about the two forms of the variational principle for the equations of motion apply here too.

The variational principles for displacement and temperature equations (11) and (13) are entirely equivalent to the field equations (2), (3) and (10), with their associated boundary conditions. The displacement and temperature functions can be obtained by simultaneously making the appropriate integrals stationary with respect to displacement and temperature respectively. The first equation (11) could be regarded as getting a stationary "cost" function, and the second (13) as a constraint, or vice versa.

AN EXAMPLE IN ONE DIMENSION

1. *The problem*

The problem of steady state longitudinal waves in a viscoelastic rod with thermomechanical coupling is now solved using the one dimensional versions of the variational principles presented in the previous section. A Rayleigh–Ritz procedure is used on the alternative forms of the variational principles. The same problem, including time dependence, was solved by Huang and Lee [11] using a finite difference approach. The results obtained here are compared with some steady state results given in [11]. The question of how convergence to the solution given in [11] depends on the number of coordinate functions used is discussed.

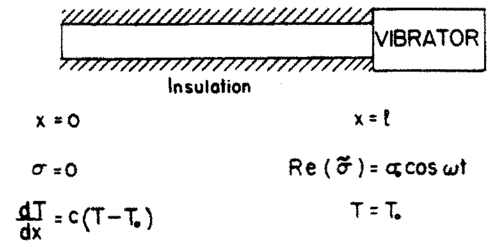

FIG. 1.

Let us consider a viscoelastic rod of length l insulated on its lateral surface as shown in Fig. 1. The left end is free while the right end is vibrated at a frequency ω with a stress amplitude σ_0 (real), so that the prescribed stress at this end is $\sigma_0 \cos \omega t$. The temperature of the vibrator is assumed constant at T_0 while a radiation boundary condition is assumed at $x = 0$.

The boundary conditions can therefore be written as

$$\begin{aligned} x = 0 \quad & \sigma = 0 \\ & \frac{dT}{dx} = c(T - T_0) \\ x = l \quad & \sigma = \sigma_0 \\ & T = T_0 \end{aligned} \tag{15}$$

where $c = h/K$ is the ratio of the surface conductance h to the thermal conductivity K of the viscoelastic material. Note that the problems of uniform normal or shear traction on the surface of a wide slab with the stated thermal boundary conditions prescribed on the slab surface are mathematically equivalent problems. Note also that here we have mixed thermal boundary conditions but this can be taken care of in the variational principle as shown later.

2. *The field equations and variational principles*

The equations of motion (2) and the constitutive equations (3) reduce to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(E^*\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\rho\omega^2 u = 0 \tag{16}$$

$$\sigma = \sigma_1 + i\sigma_2 = E^*\frac{\mathrm{d}u}{\mathrm{d}x} \tag{17}$$

where $E^* = E_1 + iE_2$ is the complex Young's modulus which is a function of the temperature through the reduced frequency (see equation 5).

The steady state energy equation becomes

$$K\frac{\mathrm{d}^2 T}{\mathrm{d}x^2}+\frac{\omega}{2}E_2\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 = 0 \tag{18}$$

where, as before,

$$\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 = \frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}.$$

Note that for the one-dimensional strain problem, E^* must be replaced by $\lambda^*+2\mu^*$ and E_2 by $\lambda_2+2\mu_2$.

The corresponding variational principle for displacement becomes (see equation 11)

$$\underset{u}{\delta}\left\{\int_0^l\left[-\frac{E^*}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2+\frac{\rho\omega^2u^2}{2}\right]\mathrm{d}x+\sigma_0 u(l)\right\} = 0. \tag{19}$$

If the admissible class of displacement functions is restricted such that the stress boundary conditions are already satisfied (through equation 17), the alternative form of the variational principle takes the simplified form

$$\int_0^l\left\{\frac{\mathrm{d}}{\mathrm{d}x}\left(E^*\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\rho\omega^2u\right\}\delta u\,\mathrm{d}x = 0. \tag{20}$$

The temperature variational principle takes the form: of all temperature distributions which satisfy $T(l) = T_0$, the temperature distribution which also satisfies the energy equation (18) and the radiation boundary condition at $x = 0$ (see equation 15) is determined from

$$\underset{T}{\delta}\left\{\int_0^l\left(\frac{1}{2}K\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)^2-\int^T\frac{\omega E_2(T')}{2}\left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2\mathrm{d}T'\right)\mathrm{d}x+h\left(\frac{T^2}{2}-TT_0\right)_{x=0}\right\} = 0. \tag{21}$$

The last term in the above expression comes about since we now have a radiation boundary condition instead of prescribed heat flux at $x = 0$.

As before, if the temperature is chosen such that both the temperature and radiation boundary conditions (at $x = l$ and at $x = 0$) are already satisfied, the temperature can be determined from

$$\int_0^l \left(K\frac{d^2T}{dx^2} + \frac{\omega E_2}{2}\left|\frac{du}{dx}\right|^2 \right) \delta T \, dx = 0. \tag{22}$$

Equations (20) and (22) are used in further calculations in this section. The object is to find the spatial distribution of temperature, displacement and then stress.

3. *The properties of the material*

A Lockheed solid propellant is an example of a thermorheologically simple material, in which, over a wide reduced frequency range, we can express the tensile compliance $D^* = 1/E^*$ by the following empirical formulae (see [8, 11])

$$\begin{aligned} D^* &= D_1 - iD_2 \\ D_1 &= c_1 \omega^\beta (T - T_1)^\gamma \\ D_2 &= c_2 \omega^\beta (T - T_1)^\gamma \end{aligned} \tag{23}$$

where c_1, c_2, β, γ, T_1 are constants.

4. *Method of solution*

The Rayleigh–Ritz procedure [13] is now used to obtain approximate solutions for the temperature and displacement (and then stress) functions from the variational equations (20) and (22).

The following dimensionless quantities are used

$$q = \frac{x}{l}, \qquad \tau = \frac{T - T_1}{T_0 - T_1}, \qquad \varkappa = cl. \tag{24}$$

Equations (20) and (22) are nonlinear and it is not possible to choose an orthogonal set of coordinate functions for the displacement and temperature. For convenience, it is assumed that the displacement and temperature distributions can be approximated by a linear combination of polynomials with coefficients to be determined. These functions are chosen such that they satisfy the boundary conditions (equation 15) for all choices of these unknown coefficients. Also, the number of terms in the series are parameters which can be set in the resulting algebraic equations for the coefficients. This enables comparison of successive approximations with the solution in [11] and thus an estimate of the degree of accuracy as a function of the number of terms taken is obtained.

The nondimensional temperature is written as

$$\tau(q) = 1 + (1-q)\left\{ b_0 + e_1 b_0 q + q^2 \sum_{i=2}^{m} b_i (1-q)^{i-2} \right\} \tag{25}$$

where $e_1 = \varkappa + 1$, m is a parameter and $b_0, b_2, b_3 \ldots b_m$ are m real constants that are to be determined ($m < 2$ implies $\sum_{i=2}^{m} \equiv 0$).

It is easily seen that the thermal boundary conditions from equation (15) are satisfied in terms of the nondimensional variables defined in equation (24).

The complex strain is written as

$$\varepsilon = \varepsilon_1 + i\varepsilon_2 = \frac{du}{dx} = q \sum_{i=0}^{n} a_i (1-q)^i \tag{26}$$

where

$$a_0 = a_0^R + ia_0^I = D^*|_{q=1}\sigma_0 = (c_1 - ic_2)\omega^\beta(T_0 - T_1)^\gamma\sigma_0$$

n is a parameter and $a_1, a_2, a_3 \ldots a_n$ are n complex constants that are to be determined.

As before, it is obvious that with this choice of strain, the stress boundary conditions from equation (15) are satisfied.

Writing $a_k = a_k^R + ia_k^I$ $(k = 1, n)$ this choice of function leads to $(2n+m)$ real unknowns which must be determined from an equal number of algebraic equations.

These nonlinear algebraic equations are now determined from the variational equations (20) and (22). Substituting the displacement and temperature expressions into equation (20) carrying out the necessary integrations and equating the coefficients of δa_j^R $(j = 1, n)$ to zero gives for $j = 1, 2, 3, \ldots, n$

$$\frac{2}{(j+1)(j+2)(j+3)} + \sum_{k=0}^{n} a_k^R d[f(j,k) - f(j,k+1)] - \sum_{k=0}^{n} \frac{(a_k^R a_0^R + a_k^I a_0^I)}{|a_0|^2}(I_{k+j} - 2I_{k+j+1} + I_{k+j+2}) = 0 \tag{27}$$

and equating the coefficients of δa_j^I $(j = 1, n)$ to zero gives for $j = 1, 2, 3, \ldots n$

$$\sum_{k=0}^{n} a_k^I d[f(j,k) - f(j,k+1)] - \sum_{k=0}^{n} \frac{(a_k^I a_0^R - a_k^R a_0^I)}{|a_0|^2}(I_{k+j} - 2I_{k+j+1} + I_{k+j+2}) = 0 \tag{28}$$

where I_p (p an integer) is a nonlinear function of $e_1, b_0, b_2, b_3, \ldots, b_m$ defined as

$$I_p = \int_0^1 \frac{(1-q)^p \, dq}{\{1 + (1-q)[b_0 + e_1 b_0 q + q^2 \sum_{i=2}^{m} b_i (1-q)^{i-2}]\}^\gamma}$$

$f(j, k)$ is a function of integers

$$f(j,k) = \frac{k(k+j+4) + (k+2)(j+3)}{(j+2)(j+3)(k+1)(k+2)(k+j+3)(k+j+4)}$$

d is a nondimensional parameter

$$d = \frac{\rho\omega^2 l^2}{\sigma_0}$$

and

$$|a_0|^2 = (a_0^R)^2 + (a_0^I)^2.$$

Note that equation (20) requires the displacement u in addition to the strain. Integration of equation (26) leads to an extra constant, say c_0, but also an extra equation obtained by equating the coefficient of δc_0 to zero. This extra constant c_0 has been eliminated from equations (27) and (28) given above.

Next, substituting the displacement and temperature expressions into equation (22) and equating the coefficient of δb_0 to zero gives

$$\frac{(3+e_1)e_1 b_0}{3} - \sum_{k=2}^{m} b_k(g_{k-1} - 2g_k + g_{k+1}) + V \sum_{i,k=0}^{n} [(a_k^R a_i^R + a_k^I a_i^I) \times \{(1+e_1)I_{i+k+1} - (2+3e_1)I_{i+k+2} + (1+3e_1)I_{i+k+3} - e_1 I_{i+k+4}\}] = 0 \tag{29}$$

and equating the coefficients of δb_j ($j = 2, m$) to zero gives for $j = 2, 3, 4 \ldots m$

$$\frac{4e_1 b_0}{j(j+1)(j+2)} - \sum_{k=2}^{m} b_k[h(j, k-1) - 2h(j, k) + h(j, k+1)]$$
$$+ V \sum_{i,k=0}^{n} (a_k^R a_i^R + a_k^I a_i^I)\{I_{i+j+k-1} - 4I_{i+j+k} + 6I_{i+j+k+1}$$
$$-4I_{i+j+k+2} + I_{i+j+k+3}\} = 0 \tag{30}$$

where g_k and $h(j, k)$ are given by

$$g_k = \frac{(k+1+e_1)(k-1)}{(k+1)}$$

$$h(j, k) = \frac{2k(k-1)}{(k+j)(k+j-1)(k+j-2)}.$$

V is the nondimensional parameter

$$V = \frac{-l^2 c_2 (T_0 - T_1)^{-\gamma-1} \omega^{1-\beta}}{2K(c_1^2 + c_2^2)}$$

and I_p has been defined before in equation (28).

Equations (27), (28), (29) and (30) constitute a set of $(2n+m)$ nonlinear algebraic equations for the $(2n+m)$ unknowns a_k^R ($k = 1, n$), a_k^I ($k = 1, n$), b_0 and b_k ($k = 2, m$).

The stresses are determined from the strains and temperature from

$$\begin{aligned} \sigma_1 &= E_1 \varepsilon_1 - E_2 \varepsilon_2 \\ \sigma_2 &= E_2 \varepsilon_1 + E_1 \varepsilon_2 \end{aligned} \tag{31}$$

and the stress at any time

$$\mathrm{Re}(\hat{\sigma}(x, t)) = \mathrm{Re}(\sigma\, e^{i\omega t}) = \sigma_1 \cos \omega t - \sigma_2 \sin \omega t. \tag{32}$$

These equations follow immediately from equation (17).

Nondimensional stresses s_1 and s_2 are defined as

$$s_1 = \lambda \sigma_1, \qquad s_2 = \lambda \sigma_2$$

and at $x = l$

$$s_0 = \lambda \sigma_0$$

where

$$\lambda = [2K\omega\rho(T_0 - T_1)]^{-\frac{1}{2}}. \tag{33}$$

5. *Results and conclusions*

Numerical calculations have been carried out for the following data for a Lockheed solid propellant [11] whose mechanical and thermal properties are qualitatively typical of many viscoelastic solids

$$c_1 = 4.61 \times 10^{-11} \,(\text{psi})^{-1}\,(\text{sec})^{\beta}\,(^{\circ}\text{F})^{-\gamma}$$
$$c_2 = 1.62 \times 10^{-11} \,(\text{psi})^{-1}\,(\text{sec})^{\beta}\,(^{\circ}\text{F})^{-\gamma}$$
$$\beta = -0.214 \qquad \gamma = 3.21$$
$$\varkappa = 1.0\dagger \qquad T_0 = 65^{\circ}\text{F}$$
$$T_1 = -125^{\circ}\text{F} \qquad l = 3 \text{ in.}$$
$$l^2\rho = 1.023 \times 10^{-4} \text{ psi-sec}^2$$
$$2K\rho(T_0 - T_1) = 8.08 \times 10^{-4} \text{ psi}^2\text{-sec}$$
$$\omega = 10^4 \text{ rad/sec.} \quad s_0 = 0.5 \;(\sigma_0 = 1.42 \text{ psi}).$$

The nonlinear algebraic equations (27–30) were solved numerically in an IBM 360/67 computer for different values of n and m. The subroutine used is given in [16]. The method is a compromise between the Newton–Raphson algorithm and the method of steepest descent. The average execution time per run was of the order of thirty seconds. Convergence was rapid and no jump instabilities of the type described by Schapery in [9] were found.

Figures 2–4 show the resulting τ, s_1 and s_2 distributions for different values of n and m and also the solution from [11] obtained by the method of finite differences. The solution for $n = 1, m = 0$ is crude but we see that the convergence to the true solution is very rapid. Figures 5–7 show the approximate solutions for $n = 4$, $m = 3$. Even with these relatively small number of terms, the stress solutions are practically identical to those given in [11],

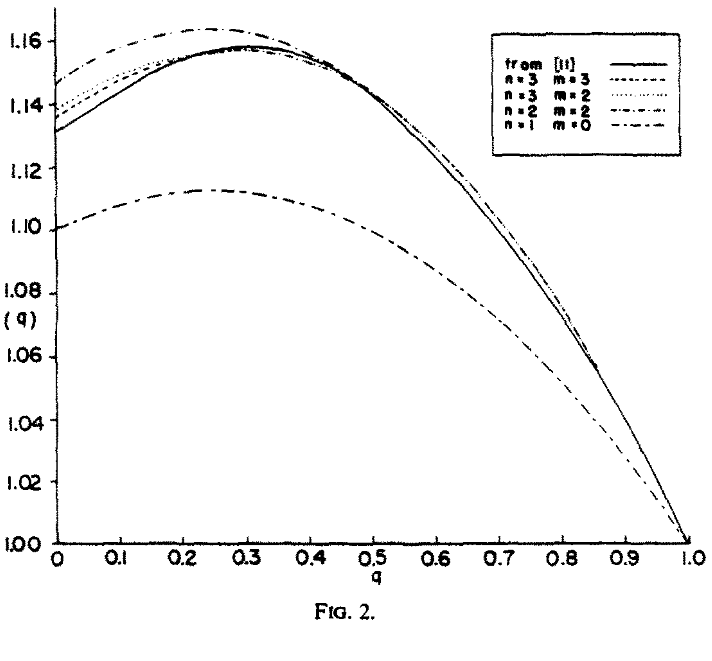

FIG. 2.

† In [11] $\varkappa$ should read 1·0 instead of 0·1.

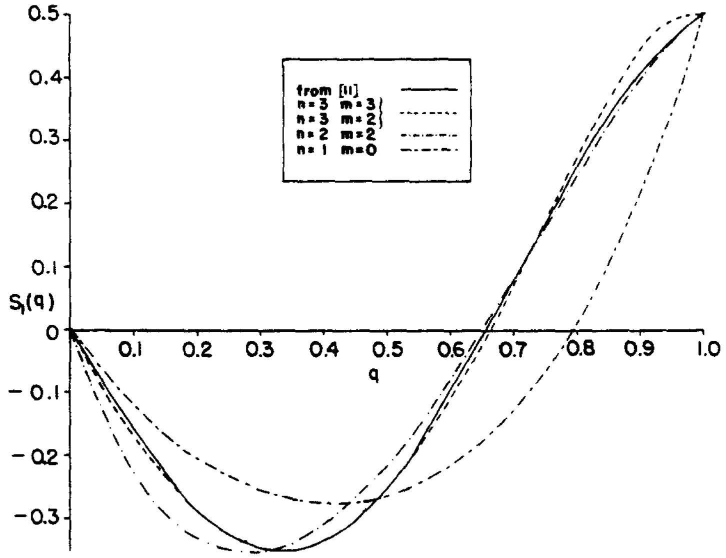

FIG. 3

while the temperature solution is well within engineering accuracy. The algorithm for solving the nonlinear algebraic equations converges very quickly and more accurate solutions can be obtained, if desired, by taking larger values of n and m.

As mentioned earlier, if the steady state values of stress and temperature are of interest (this is often the case in design), the method used here, which yields the steady state directly, is superior to that used by Huang and Lee in [11] where the complete time histories of the above mentioned quantities were determined. In some cases in [11] the authors obtained

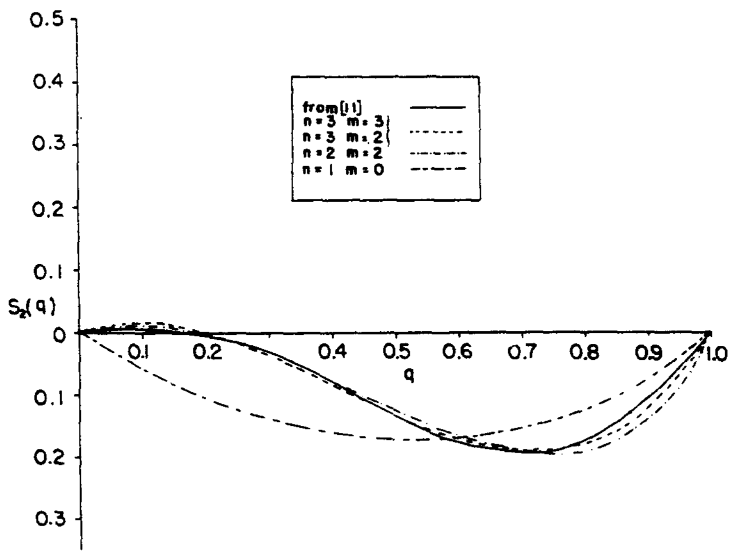

FIG. 4.

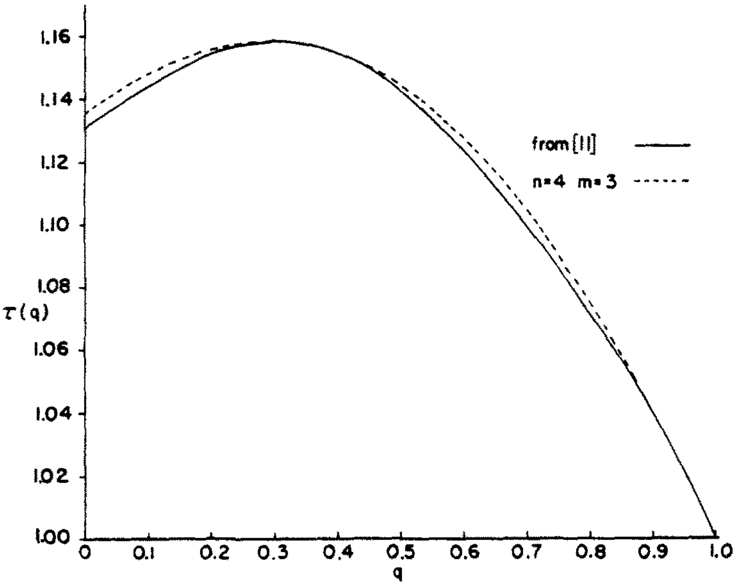

FIG. 5.

the steady state solutions by numerically integrating forward in time till the variables of interest did not change significantly. In other cases, they did not integrate up to the steady state but stopped at some large value of time.

The results obtained are thus very satisfactory as long as s_1 and s_2 are sufficiently smooth functions so that approximation by a series of polynomials is efficient. The nature of the spatial distribution of stress depends upon the particular choice of frequency and

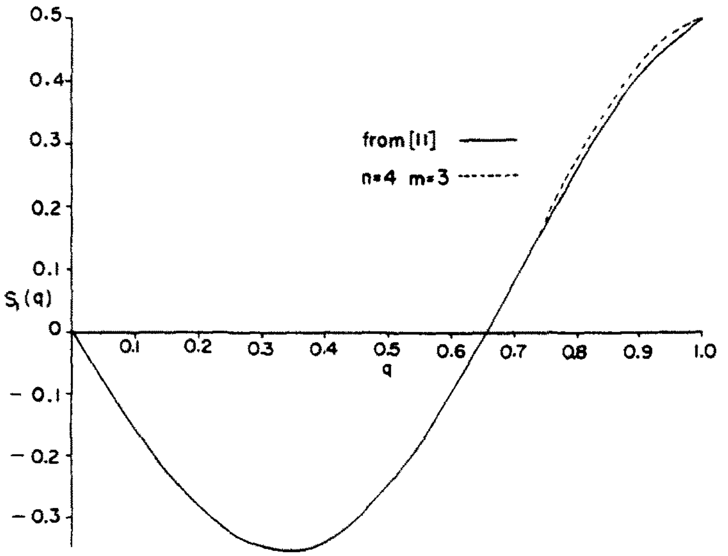

FIG. 6.

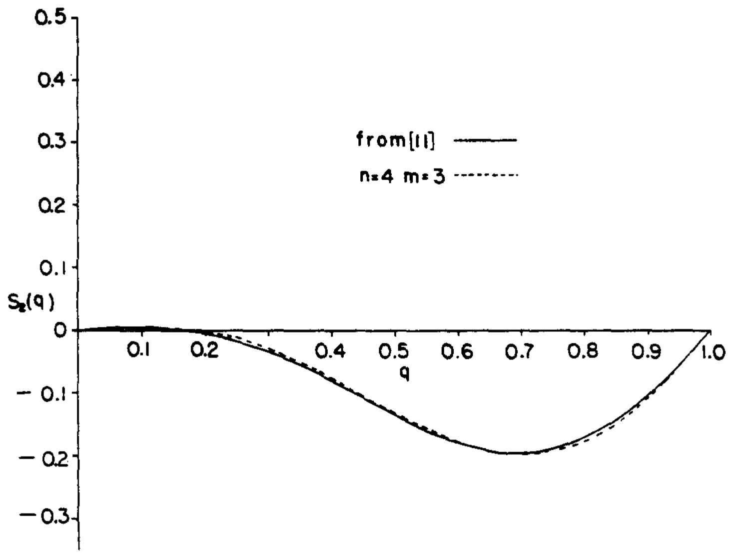

FIG. 7.

driving stress. For a given driving frequency, larger driving stresses lead to larger temperatures since more mechanical energy is dissipated as heat. This causes the material to become softer, so that lower stress wave velocities and therefore lower wave lengths result. If s_1 and s_2 are rapidly oscillating functions of q, the polynomial series is no longer efficient since a larger number of terms must be taken to get the required accuracy and the lack of orthogonality of the polynomials gives rise to Hilbert matrices. This results in convergence problems for the algorithm used to solve the algebraic equations. The variational principles, however, should work fine for these cases, if, for example, trigonometric functions are chosen instead.

To sum up, the variational approach seems comparable to the finite difference approach for waves in one dimension and ought to be more efficient in two or three dimensions where the differential equations are partial and finite difference simulation becomes much more complicated. Solving for displacements instead of stresses has the advantage of automatic satisfaction of compatibility conditions and Mitchell's equations for multiply connected regions. The choice of coordinate functions is very important and must be made carefully.

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REFERENCES

[1] M. E. GURTIN, Variational principles in the linear theory of viscoelasticity. *Arch. ration. Mech. Analysis* 13, 179 (1963).
[2] M. J. LEITMAN, Variational Principles in the Linear Dynamic Theory of Viscoelasticity. Technical Report No. 98. Division of Applied Mathematics, Brown University (1965).

[3] R. A. SCHAPERY, Irreversible Thermodynamics and Variational Principles with Applications to Viscoelasticity. Aero Research Labs, Wright Patterson Air Force Base, ARL 62-418 (1962).
[4] R. A. SCHAPERY, On the time dependence of viscoelastic variational solutions. *Quart. appl. Math.* **12**, 207 (1964).
[5] K. C. VALANIS, Exact and variational solutions to a general visco-elasto-kinetic problem. *J. appl. Mech.* **33**, 888 (1966).
[6] W. KOHN, J. A. KRUMHANSL and E. H. LEE, Variational methods for dispersion relations and elastic properties of composite materials. Paper No. 71-APMw-21. *J. appl. Mech.* (to be published).
[7] R. A. SCHAPERY, Applications of thermodynamics to thermomechanical, fracture and birefringent phenomena in viscoelastic media. *J. appl. Phys.* **35**, 1451 (1964).
[8] R. A. SCHAPERY, Effect of cyclic loading on the temperature in viscoelastic media with variable properties. *AIAA J.* **2**, 827 (1964).
[9] R. A. SCHAPERY, Thermomechanical behavior of viscoelastic media with variable properties subjected to cyclic loading. *J. appl. Mech.* **32**, 611 (1965).
[10] M. A. BIOT, *Variational Principles in Heat Transfer*. Clarendon Press (1970).
[11] N. C. HUANG and E. H. LEE, Thermomechanical coupling behavior of viscoelastic solids subjected to cyclic loading. *J. appl. Mech.* **34**, 127 (1967).
[12] R. COURANT and D. HILBERT, *Methods of Mathematical Physics*, Vol. 1. Interscience (1966).
[13] F. B. HILDERBRAND, *Methods of Applied Mathematics*. Prentice-Hall (1965).
[14] W. FLÜGGE, *Viscoelasticity*. Blaisdell (1967).
[15] I. S. SOKOLNIKOFF, *Mathematical Theory of Elasticity*. McGraw-Hill (1956).
[16] M. J. D. POWELL, A Fortran Subroutine for Solving Systems of Nonlinear Algebraic Equations, Report No. AERE-R.5947, Mathematical Branch, Theoretical Physics Div., A.E.R.E., Harwell (1968).

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Абстракт—Даются двойные вариационные принципы для стационарного распространения волн в трехразмерных термовязкоупругих средах. Первый принцип, касающийся уравнений движения, заключает только комплексную функцию перемещений. Второй служит для уравнения знергии. Эти принципы приспособливаются к случаю одноразмерного тела. На основе метода Рэлея—Ритца решается одноразмерный случай распространения волны в термовязкоупругом стержне, который излирован своей боковой поверхности и находится под влиянием синусоидального напряжения на одном конце. Функции перемещений и температуры представлены в виде рядов полиномов. Сравниваются последовательные приближения для решения, полученного выше с решением в конечных разностях. Получается оцелка степени точности, в виде функции числа членов рядов. Находится, что приближенное решение сходится быстро к точлому.